An Optimal Limiting $2D$ Sobolev Inequality [1]

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The $2D$ Sobolev embeddings

$$W^{1,p} \subset L^{\frac{2p}{2-p}}, \quad \text{for } p \in [1,2) \quad \text{and} \quad W^{1,p} \subset C^{1-2/p}, \quad \text{for } p \in (2,+\infty)$$

fail for the limiting case $p = 2$ (see e.g. [1]). In the case of a bounded domain we have the inclusion $W^{1,2} \subset L^q$ for any $q < \infty$, but not for $q = \infty$. The function $f(x) = \ln\big(1 - \min(0, \ln|x|)\big)$ gives a counter example to the limiting inclusion.

However, with a small additional regularity condition, functions in $W^{1,2}$ become bounded. The bound can be obtained by so called logarithmic Sobolev inequalities. Brezis-Gallouet [2] firstly presented this type of inequality. In particular they prove the following inequality:

$$\|u\|_{L^\infty} \leqslant C\,\|u\|_{W^{1,2}} \left(1 + \sqrt{\ln(1 + \|u\|_{W^{2,2}}/\|u\|_{W^{1,2}})}\right). \tag{1}$$

There was a number of works where similar inequalities were proven in different settings, including cases of different norms and other number of space dimensions. See, for example, [1],[2].

Inequality (1) can be rewritten as $\|u\|_{L^\infty} \leqslant \|u\|_{W^{1,2}}\, g(\|u\|_{W^{2,2}}/\|u\|_{W^{1,2}})$, where $g(t) = C\big(1 + \sqrt{\ln(1+t)}\big)$. It is natural to ask, what is the optimal (i.e., minimal) function g for this inequality. In the present article we study a similar inequality when the $W^{2,2}$ norm is replaced with the α-Hölder seminorm. We give a complete answer for the optimality question when the domain is a disk. Similar problem, however, without the optimality question was studied in [1].

Let Ω be an open domain in $\mathbb{R}^2$. Define $W_0^{1,2}(\Omega)$ to be the closure of smooth and compactly supported functions in the Sobolev space $W^{1,2}(\Omega)$.

The α-homogeneity in (1), i.e., the fact that the same function F provides the optimal inequality for all α, can be explained by the fact that the set of extremes of (1) as well as the quantities $\|\nabla u\|_{L^2}/\sqrt{\alpha}$, $\|u\|_\alpha$ and $\|u\|_{L^\infty}$ on the set of extremes are preserved under the transformation

$$u(x) \mapsto \tilde{u}(x) = u(x|x|^{\lambda-1}), \qquad \alpha \mapsto \tilde{\alpha} = \lambda\alpha,$$

provided $\alpha, \lambda\alpha \in (0,1]$.

Bibliography

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2. Girela D. *On Bloch functions and gap series* // Publications Matematiques. 1991. V. 35. P. 403–427.

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